Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 261 (2011) 3366–3389

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

A characterization of duality through section/projection correspondence in the finite dimensional setting ☆

Vitali D. Milman, Alexander Segal, Boaz A. Slomka *

School of Mathematical Science, Tel-Aviv University, Ramat Aviv, Tel Aviv, 69978, Israel

Received 20 July 2011; accepted 12 August 2011

Available online 30 August 2011

Communicated by J. Bourgain

Abstract

In this paper, we show that the well-known duality operation in the context of convex bodies in $\mathbb{R}^n$ is completely characterized by its property of interchanging sections with projections. Our results are compared to results by Böröczky–Schneider and Artstein–Milman, who showed that in many cases, the property of order reversing is sufficient to determine a duality operation, up to obvious linear modifications. In fact, we provide another result that recovers a known characterization of duality by the property of order reversing, and up to a mild condition, also a characterization of duality by interchanging sections by projections.

Keywords: Duality; Polar convex sets; Section–projection correspondence

1. Introduction

For centuries, the concept of duality has been proving out to be extremely useful for the understanding of numerous mathematical problems in various areas of mathematics. In particular, the field of convex analysis. Recently, several authors have characterized classical duality operations for many classes of convex sets and functions (see [1–4,6–8]). It was shown that

☆ The first and second named authors were partially supported by the ISF grant No. 387/09 and the third named author was partially supported by the ISF grant No. 865/07.

* Corresponding author. Fax: +972 77 4220101.

E-mail addresses: milman@post.tau.ac.il (V.D. Milman), segalale@gmail.com (A. Segal), boazslom@post.tau.ac.il, boazslomka@gmail.com (B.A. Slomka).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2011.08.007

the property of order reversing isomorphism (namely, order reversing and bijective) is often sufficient for characterization of a duality, uniquely up to obvious linear modifications.

1.1. Polarity of convex sets

In the context of convex sets, which we are focusing on in this note, we recall the following. Denote the class of all closed convex sets in $\mathbb{R}^n$ containing the origin 0 by $\mathcal{K}_0^n$, the class of all compact convex sets in $\mathbb{R}^n$ containing 0 in their interior by $\mathcal{K}_{(0),b}^n$ and let $\langle\cdot,\cdot\rangle$ denote the standard scalar product on $\mathbb{R}^n$. The classical duality operation for these classes, also known as the polarity mapping, taking each convex set K to its polar set K° is defined by the following relation

$$K^\circ = \left\{x \in \mathbb{R}^n \colon \langle x, y\rangle \leqslant 1, \ \forall y \in K\right\}.$$

It is not hard to check that this operation is an involution on each of these classes, which reverses inclusion. The following are two examples for the results mentioned above, which will be compared with the results presented here.

Theorem 1.1. *(See Böröczky and Schneider [4].) Let $n \geqslant 2$. Let $T : \mathcal{K}_{(0),b}^n \to \mathcal{K}_{(0),b}^n$ be a bijection. Assume that for every $K, L \in \mathcal{K}_{(0),b}^n$, T satisfies that $K_1 \subset K_2 \Leftrightarrow T(K_1) \supset T(K_2)$. Then, there exists a linear transformation $B \in GL(n)$ such that $TK = BK^\circ$ for every $K \in \mathcal{K}_{(0),b}^n$.*

Theorem 1.2. *(See Artstein-Avidan and Milman [2].) Let $n \geqslant 2$. Let $T : \mathcal{K}_0^n \to \mathcal{K}_0^n$ be a bijection. Assume that for every $K_1, K_2 \in \mathcal{K}_0^n$, T satisfies that $K_1 \subset K_2 \Leftrightarrow T(K_1) \supset T(K_2)$. Then, there exists a linear transformation $B \in GL(n)$ such that $TK = BK^\circ$ for every $K \in \mathcal{K}_0^n$.*

Remark 1.3. Gruber [6] has determined all endomorphisms of the lattice of unit balls of norms in $\mathbb{R}^n$. His results imply the same result as in Theorem 1.1 and Theorem 1.2 for the class of all centrally-symmetric compact sets of full dimension, which we denote by $\mathcal{K}_{(0),c}^n$. Similar results for $\mathcal{K}_{(0),b}^n$, $\mathcal{K}_0^n$ and some other classes of convex bodies can be found in [4] and [8].

The knowledge of a duality operation on a given class, with a certain interchanging property, allows one to characterize it by first determining all bijective operations with a corresponding preserving property. For example, the above results are immediately implied by the following theorem, which considers order preserving operations instead of reversing.

Theorem 1.4. *Let $n \geqslant 2$. Let $\mathcal{K}$ be either $\mathcal{K}_{(0),b}^n$, $\mathcal{K}_0^n$ or $\mathcal{K}_{(0),c}^n$. Let $T : \mathcal{K} \to \mathcal{K}$ be a bijection. Assume that $K_1 \subset K_2 \Leftrightarrow T(K_1) \subset T(K_2)$ for every $K_1, K_2 \in \mathcal{K}$. Then, there exists a linear transformation $B \in GL(n)$ such that $TK = BK$ for every $K \in \mathcal{K}$.*

Indeed, composing any order reversing isomorphism with the polarity mapping gives an order preserving isomorphism and vice versa. Since $(BK)^\circ = (B^{-1})^t K^\circ$, the desired conclusion is immediate. In the sequel, we shall formulate and prove results for maps with a "preserving" property of mapping sections to sections and formulate their duality consequences in Section 1.3.

1.2. Section–projection correspondence

Duality relation has many remarkable properties and is extremely useful construction. The fact we discussed above, that it can be uniquely recovered from such abstract property as reversing order, is surprising. At the same time, there is another property which is closely associated with duality. That is, interchanging of sections and projections of convex sets (or subspaces and quotient spaces for the case of normed spaces). Actually, this was older and more original, perhaps even ancient, understanding of duality. In this paper we formulate this approach and show that such interchange, under very mild conditions, uniquely defines duality correspondence.

One suitable object for the demonstration of this approach is the class $\mathcal{K}_0^n$, as in Theorem 1.1. Another suitable object is the family of all normed spaces of dimension at most n or, equivalently, the family of all centrally-symmetric compact convex sets in $\mathbb{R}^n$ which we denote here by $\mathcal{K}_c^n$. Note that the class $\mathcal{K}_{(0),b}^n$ (and $\mathcal{K}_{(0),c}^n$) in Theorem 1.4 is not closed under intersections with subspaces (or under projections onto subspaces) and so is not suitable for our requirements. As was explained in the end of Section 1.1, a characterization of a correspondences which map sections of convex sets to sections will lead to a characterization of duality. Let us now put the above discussion in precise form and formulate our first results.

Theorem 1.5. *Let $n \geqslant 2$. Let $T : \mathcal{K}_0^n \to \mathcal{K}_0^n$ be a bijection. Assume that for every subspace E and each $K \in \mathcal{K}_0^n$ there exists a subspace F and a subspace F' such that*

$$T(K \cap E) = T(K) \cap F, \qquad T^{-1}(K \cap E) = T^{-1}(K) \cap F'.$$

Then, there exists $A \in GL(n)$, for which $T(K) = AK$, for every $K \in \mathcal{K}_0^n$.

Note that in Theorem 1.5, the same conditions are imposed on both T and T^{-1}, in complete analogy to the results stated in Theorems 1.4, 1.2 and 1.1. This is a natural (and necessary, in last three aforementioned results) condition since the duality relation is an involution. However, in our case, for dimension $n = 3$ and higher, the same conclusion is drawn, without imposing any conditions on T^{-1}. We prove the following, stronger statements.

Theorem 1.6. *Let $n \geqslant 3$. Let $T : \mathcal{K}_0^n \to \mathcal{K}_0^n$ be a bijection. Assume that for every subspace E and each $K \in \mathcal{K}_0^n$ there exists a subspace F such that*

$$T(K \cap E) = T(K) \cap F.$$

Then, there exists $A \in GL(n)$, for which $T(K) = AK$, for every $K \in \mathcal{K}_0^n$.

Theorem 1.7. *Let $n \geqslant 3$ and $T : \mathcal{K}_c^n \to \mathcal{K}_c^n$ be a bijection. Assume that for every subspace E and each $K \in \mathcal{K}_c^n$ there exists a subspace F such that*

$$T(K \cap E) = T(K) \cap F.$$

Then, there exists $A \in GL(n)$, for which $T(K) = AK$, for every $K \in \mathcal{K}_c^n$. Moreover, if we require the same properties for the inverse of T, then the theorem holds for also for $n = 2$.

Remark 1.8. Note that in Theorems 1.5, 1.6 and 1.7 we assume that for any subspace E and each set K there exists a subspace F for which $T(K \cap E) = T(K) \cap F$. By no means do we assume that $\dim E = \dim F$ nor that F is independent of the choice of the set K. These facts are implied a posteriori.

Having two different geometric characterizations of duality, one by the property of inclusion reversing (as in Theorem 1.2) and one by the property of section–projection correspondence (as would follow from Theorem 1.6) gives rise to the question of finding a unifying theorem that would imply both characterizations. For example, is it true that a bijective mapping $T : \mathcal{K}_0^n \to \mathcal{K}_0^n$ which satisfies that $T(K \cap E) \subset T(K)$ and $T^{-1}(K \cap E) \subset T^{-1}(K)$ for each set $K \in \mathcal{K}_0^n$ and for every subspace $E \subset \mathbb{R}^n$ must be induced by a linear transformation? As a complete answer to this question is yet to be found, in this paper we show a statement which is stronger than the order reversing characterization described in Theorem 1.2, but does not imply a characterization of duality through section–projection correspondence, due to one extra condition which we were not able to dismiss. We prove the following.

Theorem 1.9. *Let $n \geqslant 3$. Let $T : \mathcal{K}_0^n \to \mathcal{K}_0^n$ be a bijection. Assume the following hold.*

(a) *For every subspace $E \subset \mathbb{R}^n$ and $K \in \mathcal{K}_0^n$, we have $T(K \cap E) \subseteq T(K)$ and $T^{-1}(K \cap E) \subseteq T^{-1}(K)$.*
(b) *There exists $x_0 \in \mathbb{R}^n$ such that for all λ_1, λ_2 if $[0, \lambda_1 x_0] \subset [0, \lambda_2 x_0]$ and $T([0, \lambda_1 x_0])$, $T([0, \lambda_2 x_0])$ are 1-dimensional sets, then $T[0, \lambda_1 x_0] \subset T[0, \lambda_2 x_0]$.*
(c) *If $s_1, s_2 \in \mathcal{K}_0^n$ and $T(s_1), T(s_2)$ are 1-dimensional sets, then s_1 and s_2 are linearly independent if and only if $T(s_1)$ and $T(s_2)$ are linearly independent.*

Then there exists $A \in GL(n)$, such that $T(K) = AK$, for every $K \in \mathcal{K}_0^n$.

Remark 1.10. A bijection $T : \mathcal{K}_0^n \to \mathcal{K}_0^n$ that preserves embedding obviously satisfies conditions (a) and (b), with (c) easily following. Hence, Theorem 1.2 follows from Theorem 1.9. However, Theorem 1.5 does not follow, at least directly, as it is obvious that conditions (a) and (c) are satisfied, but it is not clear that (b) is satisfied.

Remark 1.11. The authors do not know to prove Theorem 1.9 without requiring condition (b). Furthermore, there is evidence that such theorem might be false, as there exists a counterexample if we consider the class $\mathcal{K}_c^n$ instead of $\mathcal{K}_0^n$. The counter example is as follows. For any segment $S = [-x, x] \in \mathcal{K}_c^n$, define $T(S) = [-\frac{x}{|x|^2}, \frac{x}{|x|^2}]$ where $|\cdot|$ stands for the standard Euclidean norm on $\mathbb{R}^n$ (this is just taking the dual of S in the subspace spanned by S). For any set $K \in \mathcal{K}_c^n$ which is not a segment define

$$\lambda_K := \frac{1}{2} \min\bigl\{|K \cap E|\colon\ E \subset \mathbb{R}^n \text{ where } E \text{ is a 1-dimensional subspace}\bigr\},$$

and define $T(K) = \frac{1}{\lambda_K^2} K$. It is easy to check that T satisfies conditions (a) and (c) of Theorem 1.9, but it is not induced by a linear map. Notice that the square λ_K^2 in the definition of T ensures us that $\lambda_{T(K)} = \frac{1}{\lambda_K}$ which makes T an involution.

So far, we have presented our results using a geometric language. Using the natural correspondence between centrally-symmetric compact convex sets and normed spaces, namely identifying each normed space with its unit ball, we can formulate our results in the functional analysis language. Denote the family of all normed subspaces of $\mathbb{R}^n$ by $\mathcal{F}_n$. An element $V \in \mathcal{F}_n$ is of the form $(W, \|\cdot\|)$ where W is a subspace of $\mathbb{R}^n$ and $\|\cdot\|$ is some norm on W. For a given subspace E of $\mathbb{R}^n$, the intersection with elements of $\mathcal{F}_n$ is defined as follows

$$V \cap E = \big(W \cap E, \|\cdot\|_{W\cap E}\big).$$

Then the following fact is equivalent to Theorem 1.7:

Theorem 1.12. *Let $n \geqslant 3$. Let $T : \mathcal{F}_n \to \mathcal{F}_n$ be a bijection. Assume that for every $V \in \mathcal{F}$ and any subspace $E \subset \mathbb{R}^n$ there exists a subspace $F \subset \mathbb{R}^n$ so that*

$$T(V \cap E) = T(V) \cap F.$$

Then, there exists a linear transformation $G \in GL(n)$ such that for every $(V, \|\cdot\|) \in \mathcal{F}$

$$T\big(V, \|\cdot\|\big) = \big(GV, \big\|G^{-1}\cdot\big\|\big).$$

Remark 1.13. Note that a priori, $\dim E$ and $\dim F$ might be different, but a posteriori we see that they may be selected equal.

1.3. Duality results

In this part, we formulate the characterization of duality through the mentioned property of interchanging sections with projections. We fixed above a Euclidean structure $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^n$, and we denote the orthogonal projection onto a subspace E, by $P_E(\cdot)$. We write $\|\cdot\|^\star$ for the dual norm to a norm $\|\cdot\|$. The following is a consequence of Theorem 1.6.

Corollary 1.14. *Let $n \geqslant 3$. Let $T : \mathcal{K}_0^n \to \mathcal{K}_0^n$ be a bijection. Assume that for every subspace E and each $K \in \mathcal{K}_0^n$, there exists a subspace F such that*

$$T(K \cap E) = P_F\big(T(K)\big) + F^\perp.$$

Then there exists $A \in GL(n)$, such that $T(K) = AK^\circ$, for every $K \in \mathcal{K}_0^n$. If, additionally, we require that the inverse of T has the same properties, in particular if T is an involution, then the claim holds for $n = 2$.

As the class $\mathcal{K}_c^n$ is not closed under the duality operation $K \to K^\circ$, we need a slightly modified version of the duality operation in order to formulate the dual statement of Theorem 1.7; For each set $K \in \mathcal{K}_c^n$, define its dual set with respect to the subspace $\operatorname{sp} K := \mathit{span}\{x\colon x \in K\}$ by

$$K^\star = \big\{x \in \operatorname{sp} K\colon \langle x, y\rangle \leqslant 1,\ \forall y \in K\big\}. \tag{1}$$

This way, we make sure that $K^\star$ is of the same dimension as K and that the class $\mathcal{K}_c^n$ is closed under this operation. Also, notice that the dual norm $\|\cdot\|^\star$ is induced by the convex set which is the $\star$-polar of the unit ball of the relevant subspace. Now, we are ready to formulate the dual version of Theorem 1.7.

Corollary 1.15. *Let $n \geqslant 3$. Let $T : \mathcal{K}_c^n \to \mathcal{K}_c^n$ be a bijection. Assume that for every subspace E and each $K \in \mathcal{K}_c^n$, there exists a subspace F such that*

$$T(K \cap E) = P_F\big(T(K)\big).$$

Then there exists $A \in GL(n)$, such that $T(K) = AK^\star$, for every $K \in \mathcal{K}_c^n$. As above, if the inverse of T satisfies the same properties, the claim holds for $n = 2$.

We may reformulate duality results in the language of Functional Analysis. It is useful to keep our fixed scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$ as the fixed duality relation also for subspaces of $\mathbb{R}^n$. Let E be a normed subspace with the unit ball B_E. Then, for a subspace $F \subset E$, the quotient space E/F is a normed space with the unit ball $P_{F^\perp}(B_E)$. Let us formulate the way we "project" elements of $\mathcal{F}_n$ on a subspace F. Consider an element $V = (E, \|\cdot\|) \in \mathcal{F}_n$ and a subspace $F \subset \mathbb{R}^n$. Then define

$$V/F := \big(E/(E \cap F), \|\cdot\|_{E/(E\cap F)}\big).$$

We have the following formulation of the dual to Theorem 1.12:

Corollary 1.16. *Let $n \geqslant 3$. Let $T : \mathcal{F}_n \to \mathcal{F}_n$ be a bijection. Assume that for every $V \in \mathcal{F}_n$ and any subspace $E \subset \mathbb{R}^n$ there exists a subspace $F \subset \mathbb{R}^n$ so that*

$$T(V \cap E) = T(V)/F.$$

Then, there exists a linear transformation $G \in GL(n)$ such that for all $(V, \|\cdot\|) \in \mathcal{F}_n$,

$$T\big(V, \|\cdot\|\big) = \big(GV^*, \big\|G^{-1}\cdot\big\|^*\big).$$

The paper is organized as follows. In Section 2 we prove Theorem 1.6. In Section 3, we fill in the missing details for the proof of Theorem 1.5 in the case of dimension $n = 2$. For higher dimensions, Theorem 1.5 is implied by Theorem 1.6. In Section 4, we discuss the necessary adjustments to the arguments in the proof of Theorem 2, in order to prove Theorem 1.7. In Section 5, Theorem 1.9 is proven. In addition, we add a short Appendix A with one result on order preserving mappings on the Grassmannian, which is in the spirit of the main results of this paper but does not formally follow from them.

2. Proof of Theorem 1.6

The proof is composed of several steps. Before we start the detailed proof we will describe the general plan. At first we will show that for any dimension $k \leqslant n$, T maps k-dimensional sets onto k-dimensional sets, and that subspaces are mapped to subspaces. As a consequence, we will be able to prove that pre-images of linearly independent segments are linearly independent segments. Our next goal, which perhaps is more involved, is showing that pre-images of

1-dimensional sections of a set $K \in \mathcal{K}_0^n$ are 1-dimensional sections of its pre-image $T^{-1}(K)$. Finally, we will investigate the properties of 1-dimensional sets, under our mapping, and will show that it is actually an order-isomorphism of $\mathcal{K}_0^n$, which by known results, is induced by a linear transformation.

We will use the following notation. We will call a segment of the form $[0, x]$ a *decent segment* and denote it by $\bar{x}$. A ray in direction $u \in \mathbb{R}^n$ will be denoted by R_u. The closure of the convex hull of $K_1, K_2 \in \mathcal{K}_0^n$ will be denoted by $K_1 \vee K_2$. The dimension of a set K, denoted by $\dim K$ is the dimension of the subspace spanned by K.

2.1. *Fixed origin*

Lemma 2.1. $T(\{0\}) = \{0\}$.

Proof. If $T(\{0\}) \neq \{0\}$, then $T(K_0) = \{0\}$ for some set $K_0 \neq \{0\}$, due to the surjectivity of T. We may choose a 1-dimensional subspace E for which $K_0 \cap E \neq K_0$ and so there exists a subspace F such that $T(K_0 \cap E) = T(K_0) \cap F = 0$, which is a contradiction to the injectivity of T. $\square$

2.2. *Dimension invariance*

Proposition 2.2. $\dim K = \dim T(K)$ *for all* $K \in \mathcal{K}_0^n$.

In order to prove Proposition 2.2, we need the following lemma. The proof of the proposition will follow the proof of the lemma.

Lemma 2.3. *Let* $K \in \mathcal{K}_0^n$ *be a set of dimension* $m \leqslant n$. *Then* $T(K)$ *is of dimension greater or equal to* m.

Proof. By induction. We have shown before that $\{0\}$ is mapped to $\{0\}$, thus 1-dimensional sets must be mapped to sets of dimension at least 1, so we have the base of our induction. Assume the claim holds for all sets of dimension less or equal to $m - 1$. Let $K \in \mathcal{K}_0^n$ be an m-dimensional set and assume that $T(K)$ is p-dimensional, where $p < m$. We may choose a p-dimensional subspace E_p such that $E_p \cap K$ is again p-dimensional. Since T maps sections of sets to sections of their images, we have $T(K \cap E_p) = T(K) \cap F$ for some subspace F. By the induction hypothesis we know that the set $K \cap E_p$ is mapped to a set of dimension at least p. Since $T(K)$ is p-dimensional, it follows that $T(K \cap E_p) = T(K) \cap F = T(K)$, which is a contradiction to injectivity of T. $\square$

Proof of Proposition 2.2. First, let us check that pre-images of n-dimensional sets are n-dimensional. Assume that the pre-image of an n-dimensional set $\tilde{K} \in \mathcal{K}_0^n$ is a k-dimensional set K, with $k < n$. Then, there exists a vector v, orthogonal to the subspace spanned by K. Consider the set $K' = K \vee v$. Then, there exists a subspace F for which $T(K) = T(K' \cap \operatorname{sp} K) = T(K') \cap F$. Since $T(K)$ is n-dimensional, it follows that $F = \mathbb{R}^n$ and so $T(K) = T(K')$, a contradiction to the injectivity of T. The fact that pre-images of sets of dimension $m < n$ are also m-dimensional follows by induction, using the same argument. Indeed, assume that the claim holds for all sets of dimension $> m$ and let $K \in \mathcal{K}_0^n$ for which $T(K)$ is m-dimensional. Suppose that $\dim K \neq m$. Then, by Lemma 2.3 we have $\dim K < m$. Choose a vector v orthogonal to $\operatorname{sp} K$ and define $K' = K \vee v$. Then, there exists a subspace F for which $T(K) = T(K' \cap \operatorname{sp} K) =$

$T(K') \cap F$. Since $T(K)$ is of dimension m, we must have that $T(K')$ is of dimension at least m. On the other hand, since $\dim K' \leqslant m$, our induction hypothesis implies that $\dim T(K') \leqslant m$. Thus $\dim T(K') = m$, which means that $T(K) = T(K')$. This is a contradiction to the injectivity of T. $\square$

2.3. *Linearly independent segments, part I – The image*

Throughout this note, 1-dimensional sets (finite or infinite) in $\mathcal{K}_0^n$ will also be referred to as segments.

Lemma 2.4. *Linearly independent segments are mapped to linearly independent segments.*

Proof. Assume we have two linearly independent segments S_1, S_2 and define the set $K = conv\{S_1, S_2\}$. Then, by Proposition 2.2, for $i = 1, 2$, $T(S_i) = T(K \cap \operatorname{sp} S_i) = T(K) \cap L_i$, for some 1-dimensional subspace L_i. Since T is injective, $L_1 \neq L_2$, otherwise we would have that $T(S_1) = T(S_2)$. Hence $T(S_1)$ and $T(S_2)$ are linearly independent. $\square$

Later on, in Lemma 2.6 we will show that the opposite direction also holds. That is, that pre-images of linearly independent segments are linearly independent.

2.4. *Subspace invariance*

Lemma 2.5. *Let $E \in \mathcal{K}_0^n$ be a **subspace** of $\mathbb{R}^n$ of dimension $m \leqslant n$. Then, $T(E)$ and $T^{-1}(E)$ are subspaces of dimension m.*

Proof. The fact that T preserves the dimension of any set was already proven in Proposition 2.2. So we only need to make sure that both the image and the pre-image of any subspace are also subspaces. This is done by induction on the dimension m. As the induction base, let us prove that the statement holds for $m = 1$. First, we will show that a segment which is not a subspace cannot be mapped to a subspace. Let L be a segment which is not a subspace and assume that $T(L)$ is a 1-dimensional subspace. Choose a segment L' which is linearly independent of L and for which $T(L')$ is a 1-dimensional subspace. Indeed, such a segment exists since we can take any 2-dimensional set M which is mapped to a 2-dimensional subspace and take L' to be one of its sections. Now, we need to take care of two cases:

Case 1. Assume that L' is not a subspace. Take $K := L \vee L'$. Since L and L' are linearly independent sections of K and both $T(L)$ and $T(L')$ are 1-dimensional subspaces, it follows that $T(K)$ is the 2-dimensional subspace which is spanned by $T(L)$ and $T(L')$. On the other hand, we could take another 2-dimensional set K' for which both L and L' are its sections and so by the same reasoning, we would get that $T(K') = T(K)$, which contradicts the fact that T is injective.

Case 2. L' is a subspace. Notice that in this case the set K is a slab that is mapped to a 2-dimensional subspace. Take any 1-dimensional section S of K that is neither L nor L'. Then, $T(S)$ must be a section of $T(K)$ which implies that $T(S)$ is a subspace (but S is not). Define $M := L \vee S$. Since $T(L)$ and $T(S)$ are subspaces, we have $T(M) = T(K)$ which again contradicts injectivity.

Next, we show that 1-dimensional subspaces are mapped to 1-dimensional subspaces. Let $E \in \mathcal{K}_0^n$ be a 1-dimensional subspace and assume that $T(E)$ is not a subspace. Denote the pre-image of $\operatorname{sp} T(E)$ by L. By Lemma 2.4, E and $T^{-1}(L)$ are linearly dependent, and so, by injectivity, L cannot be a subspace. This contradicts our previous argument, and so $T(E)$ is a 1-dimensional subspace. This completes the proof of the induction base.

The induction step for $m > 1$ is done by taking sections. Assume that the lemma holds for any dimension $p < m$. Let $K \in \mathcal{K}_0^n$ be an m-dimensional set which is not an m-dimensional subspace. Take a 1-dimensional section S of K which is not a subspace. By the induction hypothesis, $T(S)$ is not a subspace which means that $T(K)$ cannot be a subspace either. In order to show that a subspace is mapped to a subspace, take an m-dimensional subspace $E \in \mathcal{K}_0^n$, and two distinct $(m-1)$-dimensional sections L_1, L_2 of E. By the induction hypothesis, $T(L_1)$ and $T(L_2)$ are two distinct $(m-1)$-dimensional subspaces which are contained in $T(E)$ (as they are sections of $T(E)$). Thus $T(E)$ contains $T(L_1) \vee T(L_2)$, which is a subspace of dimension at least m. Since $T(E)$ is of dimension m (dimension is preserved, by Proposition 2.2), it follows that it is a subspace, as required. □

2.5. Linearly independent segments, part II – The pre-image

Lemma 2.6. *The pre-images of linearly independent segments are linearly independent segments.*

Proof. Let $E \in \mathcal{K}_0^n$ be a 1-dimensional subspace and let $L \subset E$ be a segment. It is enough to show that $T(L) \subset T(E)$. Indeed, since $T(L)$ and $\operatorname{sp} T(L)$ are linearly dependent, Lemma 2.4 implies that $T^{-1}(\operatorname{sp} T(L))$ and L both lie in E. Moreover, Lemma 2.5 implies that $T^{-1}(\operatorname{sp} T(L))$ is a 1-dimensional subspace, and so must be E itself. Thus $T(L) \subset \operatorname{sp} T(L) = T(E)$. □

Lemma 2.7. *Let $E \in \mathcal{K}_0^n$ be a subspace. Then for any set $K \in \mathcal{K}_0^n$ for which $K \subset E$, we have that $T(K) \subset T(E)$. Moreover, as a consequence we have that* $\operatorname{sp} T(K) = T(\operatorname{sp} K)$ *for every $K \in \mathcal{K}_0^n$.*

Proof. First, notice that for any two subspaces $E_1 \subset E_2$ we have $T(E_1) \subset T(E_2)$, since $T(E_1) = T(E_1 \cap E_2) = T(E_2) \cap F$ for some subspace F.

Now, we turn to prove the lemma by induction on the dimension of K. Assume $K \in \mathcal{K}_0^n$ is 1-dimensional and that $K \subset E$ for some subspace E. Then, $K \subset E_1$ where $E_1 = \operatorname{sp} K \subset E$. By Lemma 2.6 we know that K and E_1 are linearly dependent segments. Lemma 2.5 implies that $T(E_1)$ is a 1-dimensional subspace and so $T(K) \subset T(E_1)$. As we showed that $T(E_1) \subset T(E)$, we have $T(K) \subset T(E)$.

Assume that our claim holds for sets of dimension less than m and assume that $K \subset E$ where K is m-dimensional and E is a subspace. Take two distinct $(m-1)$-dimensional sections of K, S_1, S_2. Then, $T(S_1)$ and $T(S_2)$ are two distinct $(m-1)$-dimensional sections of $T(K)$ and so $T(S_1) \vee T(S_2) \subset T(K)$ where both $T(S_1) \vee T(S_2)$ and $T(K)$ are of dimension m. By the induction hypothesis we have that $T(S_1) \subset T(E)$ and $T(S_2) \subset T(E)$ and so $T(S_1) \vee T(S_2) \subset T(E)$. Since, by Lemma 2.5, $T(E)$ is a subspace and since $T(K)$ and $T(S_1) \vee T(S_2)$ lie in the same m-dimensional subspace, it follows that $T(K) \subset T(E)$.

Now, we prove that $\operatorname{sp} T(K) = T(\operatorname{sp} K)$ for every $K \in \mathcal{K}_0^n$. Indeed, by the above argument we have that $T(K) \subset T(\operatorname{sp} K)$. By Proposition 2.2, both $T(K)$ and $T(\operatorname{sp} K)$ are of the same dimension, and since, by Lemma 2.5 $T(\operatorname{sp} K)$ is a subspace and so $\operatorname{sp} T(K) = T(\operatorname{sp} K)$. □

2.6. 1-Dimensional sections

Our next goal is to show that T shares its properties with T^{-1}. More precisely, it will suffice for us to show that pre-images of 1-dimensional sections of any given set are sections of its pre-image. Notice that so far we have shown that pre-images of 1-dimensional sections may be either sections of the given set, or they must have zero intersection with this set. It is left to show that the latter cannot happen. To this end, we would like to show first an important, and a priori unclear fact: for each 1-dimensional subspace E there exists a subspace E' so that for any $K \in \mathcal{K}_0^n$, $T(K \cap E) = T(K) \cap E'$. That is, that the corresponding subspace E' is independent of the choice of K. Since we already know that $T(E)$ is a 1-dimensional subspace, we will actually show that $T(E)$ satisfies this condition.

2.6.1. 1-Dimensional intersection

The first, simple case, is when the intersection of a set with the 1-dimensional subspace is 1-dimensional.

Lemma 2.8. *Let $E \subset \mathbb{R}^n$ be 1-dimensional subspace, and assume that for a given set $K \in \mathcal{K}_0^n$, $\dim(K \cap E) = 1$. Then, $T(K \cap E) = T(K) \cap T(E)$.*

Proof. The claim immediately follows from the fact that linearly dependent segments are mapped to linearly dependent segment, which was proven in Lemma 2.6, and the fact that $T(E)$ is a 1-dimensional subspace, which was proven in Lemma 2.5. □

2.6.2. Zero intersection

The second, more delicate case, we are left to consider is when the intersection of a set K with the 1-dimensional subspace E is exactly $\{0\}$. Taking care of this case requires several steps. In the next step we deal with the case in which E does not belong to $\operatorname{sp} K$.

Lemma 2.9. *Let $K \in \mathcal{K}_0^n$ be a set of dimension $m < n$. Then, for any 1-dimensional subspace E for which $\operatorname{sp} K \cap E = \{0\}$, we have that $\operatorname{sp} T(K) \cap T(E) = \{0\}$. In particular, $T(K \cap E) = T(K) \cap T(E) = \{0\}$.*

Proof. Let E be a 1-dimensional subspace that does not intersect $\operatorname{sp} K$ and assume its image does intersect $\operatorname{sp} T(K)$ (which means that $T(E) \subset \operatorname{sp} T(K)$). Define the cylinder $\tilde{K} = K \vee E$. It is mapped to a set of which both $T(K)$ and $T(E)$ are its sections, which is possible only if $E' = T(E) \subset T(K)$. However, if this were the case, we could choose some finite segment $S \subset E$ and define a different cylinder $K' = K \vee S$. The image of K' would have both $T(K)$ and $T(S)$ as its sections and so, on the one hand, $T(E) \subset T(K) \subset T(K')$. On the other hand, Lemma 2.8 implies that $T(S) = T(K' \cap E) = T(K') \cap T(E) = T(E)$, which would contradict the injectivity of T. □

As a corollary from the preceding argument we get the reverse direction of Lemma 2.7:

Corollary 2.10. *Let $E \subset \mathbb{R}^n$ be a subspace. Then for any $K \in \mathcal{K}_0^n$ such that $K \subset E$ we have $T^{-1}(K) \subset T^{-1}(E)$.*

Indeed, if the claim were false, we could find a 1-dimensional section L of K such that its pre-image has 0-intersection with $T^{-1}(E)$, which would contradict Lemma 2.9.

In the following steps, we deal with the case in which our 1-dimensional subspace does belong to $\operatorname{sp} K$.

Lemma 2.11. *Let $K \in \mathcal{K}_0^n$ be a 2-dimensional set and $E \subset \operatorname{sp} K$ a 1-dimensional subspace such that $K \cap E = 0$. Let $M \in \mathcal{K}_0^n$ be a* **compact** *2-dimensional set such that $M \cap E = 0$ and $\operatorname{sp} M \cap \operatorname{sp} K = E$. Then, K and M are sections of $K \vee M$.*

Proof. It is enough to show that K is a section of $K \vee M$. Assume that there exists a point z in $\operatorname{sp} K$, such that $z \notin K$, but $z \in K \vee M$. This means that there exist two sequences of points $\{p_n\} \subset K$, $\{q_n\} \subset M$, and a sequence $\{\lambda_n\} \subset [0, 1]$ such that $z_n := \lambda_n p_n + (1 - \lambda_n)q_n$ converges to z. Due to compactness we may assume that $\{\lambda_n\}$ converges to some $0 \leqslant \lambda \leqslant 1$ and $\{q_n\}$ converges to some $q \in M$. Thus, we get that $z = \lim(\lambda_n p_n) + (1 - \lambda)q$. Since both z and $\lambda_n p_n$ belong to $\operatorname{sp} K$, we conclude that $q \in \operatorname{sp} K$ and so $q \in \operatorname{sp} K \cap M$. Since $\operatorname{sp} K \cap M = \{0\}$, it follows that $q = 0$. Thus, $z = \lim(\lambda_n p_n)$, which implies that $z \in K$, a contradiction. □

Lemma 2.12. *Let $K \in \mathcal{K}_0^n$ such that $\dim K = 2$. Then, for any 1-dimensional subspace $E \subset \operatorname{sp} K$ such that $E \cap K = 0$ we have $T(K) \cap T(E) = 0$.*

Proof. Assume that the claim does not hold and $T(E)$ has 1-dimensional intersection L with $T(K)$. Choose a 2-dimensional subspace F, such that $F \cap \operatorname{sp} K = E$. By Lemma 2.7 and Lemma 2.8, it follows that $T(F) \cap \operatorname{sp} T(K) = T(E)$.

Assume first that K is compact. Choose a 2-dimensional set $M \subset T(F)$, such that $T(E) \cap M = 0$. By Corollary 2.10, $T^{-1}(M) \subset F$. Also notice that $T^{-1}(M) \cap E = 0$ because otherwise we could apply Lemma 2.8 and so would have that $T(T^{-1}(M) \cap E) = M \cap T(E) \neq 0$ which would be a contradiction to Lemma 2.1. Thus, we see that K, $T^{-1}(M)$ and E satisfy conditions of Lemma 2.11. Hence, K and $T^{-1}(M)$ are sections of $K' := K \vee T^{-1}(M)$. By properties of T this implies that $T(K)$ and M are sections of $T(K')$, which is impossible since the section $T(K') \cap \operatorname{sp} M$ equals M but on the other hand both $L \subset \operatorname{sp} M = T(F)$ and $L \subset T(K) \subset T(K')$ and so $L \subset T(K') \cap \operatorname{sp} M = M$, which is a contradiction.

If K is not compact, choose a 2-dimensional compact set $N \subset F$, such that $N \cap E = 0$. By the previous step, $T(N) \cap T(E) = 0$. On the other hand, using again Lemma 2.11, we have that N and K are sections of $\tilde{K} := N \vee K$. Hence, $T(N)$ is a section of $T(\tilde{K})$ which is a contradiction again since it does not contain L, where $L \subset \operatorname{sp} M = T(F)$. □

Remark 2.13. Notice that the proof of Lemma 2.12 is the only place where we use a construction that requires the dimension to be at least 3.

We are finally ready to conclude the main result of this section:

Proposition 2.14. *The following statements hold*:

(a) *Let $K \in \mathcal{K}_0^n$ and let $E \in K_0^n$ be a 1-dimensional subspace. Then, $T(K \cap E) = T(K) \cap T(E)$.*
(b) *Let $K \in \mathcal{K}_0^n$. Then, the pre-image of every 1-dimensional section of $T(K)$ is a 1-dimensional section of K.*

Proof. First we prove statement (a). If $\dim(K \cap E) = 1$, the statement follows from Lemma 2.8. Assume that $K \cap E = \{0\}$. Choose a segment $L \subset K$ ($L \in \mathcal{K}_0^n$) and denote the 2-dimensional subspace spanned by L and E by V. Consider the section $\mathcal{K}' := K \cap V$ of K. We claim that $T(K' \cap E) = T(K') \cap T(E) = \{0\}$. Indeed, if K' is a segment, the claim follows from Lemma 2.4 and if K' is 2-dimensional then the claim follows from Lemma 2.12. Now, we have that

$$T(K \cap E) = T\bigl(K' \cap E\bigr) = T\bigl(K'\bigr) \cap T(E) = \bigl[T(K) \cap \operatorname{sp} T\bigl(K'\bigr)\bigr] \cap T(E).$$

By Lemma 2.7, $T(E) \subset \operatorname{sp} T(K')$ and so $T(K \cap E) = T(K) \cap T(E)$, as required.

Next, we prove statement (b). Let $K \in \mathcal{K}_0^n$. Then, the pre-image of every 1-dimensional section of $T(K)$ is a 1-dimensional section of K. Let S be a 1-dimensional section of K and denote the pre-image of $\operatorname{sp} S$ by E, i.e., $T(E) = \operatorname{sp} S$. By Lemma 2.5, E is a 1-dimensional subspace. Now, by Proposition 2.14(a), $T(T^{-1}(K) \cap E) = K \cap T(E) = S$. $\square$

2.7. Order-isomorphism

In this last section, we shall prove that T is an order-isomorphism and so, by Theorem 1.4, it is induced by a linear point map on $\mathbb{R}^n$. First, we need the following steps.

Recall that a segment is called decent if 0 is one of its endpoints.

Lemma 2.15. *Let $S \in \mathcal{K}_0^n$ be a 1-dimensional set. Then, S is a decent segment if and only if $T(S)$ is a decent segment.*

Proof. Let S be a decent segment. Take a decent segment L linearly independent of S and consider $K := S \vee L$. Obviously, there exists 1-dimensional subspace $E \subset \operatorname{sp} K$ such that $E \cap K = 0$. By Proposition 2.14(a) we know that $T(K \cap E) = T(K) \cap T(E) = 0$. Thus, 0 is on the boundary of K and every 1-dimensional section of $T(K)$ (in particular $T(S)$) is a decent segment.

The implication in the other direction follows in the same way; Let $T(S)$ be a decent segment. Take a decent segment L, linearly independent of $T(S)$ and consider $K := T(S) \vee L$. Again, there exists a 1-dimensional subspace $E \subset \operatorname{sp} K$, such that $E \cap K = 0$, hence $T^{-1}(K) \cap T^{-1}(E) = 0$ which means that 0 is on the boundary of $T^{-1}(K)$. By Proposition 2.14(b) we know that S is a section of the pre-image of K, which means that it is a decent segment. $\square$

Lemma 2.16. *Let $[0, a]$ and $[0, b]$ be segments in opposite directions. Then, $T([0, a])$ and $T([0, b])$ are decent segments in opposite directions. Similarly, $T^{-1}([0, a])$ and $T^{-1}([0, b])$ are decent segments in opposite directions.*

Proof. By Proposition 2.2, dimension is preserved under T. By Lemma 2.6 we have that $T([0, a])$ and $T([0, b])$ are linearly dependent. Let $[0, x]$ be a segment which is linearly independent of $[0, a]$ (and of $[0, b]$). Then, Lemma 2.4 implies that $T([0, x])$ is linearly independent of $T([0, a])$ (and of $T([0, b])$). Moreover, Lemma 2.15 implies that $T([0, x])$, $T([0, a])$ and $T([0, b])$ are all decent segments.

Assume that $T([0, a])$ and $T([0, b])$ are not in opposite directions. Define the sets $K_1 = [0, x] \vee [0, a]$ and $K_2 = [0, x] \vee [0, b]$. Then, there exist subspaces F_1 and F_2 for which $T([0, x]) = T(K_1 \cap \operatorname{sp} x) = T(K_1) \cap F_1$ and $T([0, a]) = T(K_1 \cap \operatorname{sp} x) = T(K_1) \cap F_2$. In other words, $T([0,x])$ and $T([0,a])$ are sections of $T(K_1)$. Similarly, $T([0,x])$ and $T([0,b])$ are sections of $T(K_2)$. Thus, we have that $T(K_1) \supset T([0,x]) \vee T([0,a])$ and $T(K_2) \supset T([0,x]) \vee T([0,b])$.

Let E be a 1-dimensional subspace for which $E \cap (T([0,x]) \vee T([0,a])) \neq \{0\}$ and $E \cap T([0,x]) = \{0\}$. Then, we also have that $E \cap T(K_1) \neq \{0\}$ and $E \cap T(K_2) \neq \{0\}$. By Proposition 2.14(b), $T^{-1}(E)$ is a subspace for which $T^{-1}(E \cap T(K_1)) = K_1 \cap T^{-1}(E) \neq \{0\}$ and $T^{-1}(E \cap T(K_2)) = K_2 \cap T^{-1}(E)$, which are both different from $\{0\}$ due to the injectivity of T and the fact that $T(\{0\}) = \{0\}$ (which was proved in Lemma 2.1). As, by our construction, $\operatorname{sp} x$ is the only 1-dimensional subspace for which both $K_1 \cap F \neq \{0\}$ and $K_2 \cap F \neq \{0\}$, it follows, for example, that $T([0,x]) = T(K_1 \cap \operatorname{sp} x) = T(K_1) \cap E$. But, we chose E so that $E \cap T([0,x]) = \{0\}$, a contradiction.

In the exact same manner, we show that $T^{-1}([0,a])$ and $T^{-1}([0,b])$ are in opposite directions (only interchanging the roles of Lemma 2.4, Lemma 2.6, Lemma 2.15 and the fact that T maps sections of a set into its sections, with Lemma 2.6, Lemma 2.4, Lemma 2.15 and Proposition 2.14(b), respectively). □

Lemma 2.17. *Let $[a,b] \in \mathcal{K}_0^n$ be a segment of the form $[a,b] = [0,a] \vee [0,b]$. Then $T([a,b]) = T([0,a]) \vee T([0,b])$ and $T^{-1}([a,b]) = T^{-1}([0,a]) \vee T^{-1}([0,b])$.*

Proof. By Lemma 2.16, $T([0,a])$ and $T([0,b])$ are decent segments in opposite directions. Let $[0,x]$ be linearly independent of $[a,b]$. Define $K_1 := [0,a] \vee [0,x]$, $K_2 := [0,b] \vee [0,x]$ and $K_3 := [a,b] \vee [0,x]$. Each 1-dimensional section of K_1 (different from $[0,a]$) is also a section of K_3 and hence, Proposition 2.14(b) implies that $T(K_1) \subset T(K_3)$. Indeed, the pre-image of any 1-dimensional section S of $T(K_1)$ (except $T([0,a])$) is a section of K_1 which is also a section of K_3 and so its image S is a section of $T(K_3)$. This implies that also $T([0,a]) \subset T(K_3)$. Thus $T(K_1) \subset T(K_3)$. Similarly, $T(K_2) \subset T(K_3)$, which means that $T(K_1) \vee T(K_2) \subset T(K_3)$. On the other hand, each 1-dimensional section of K_3 (different from $[a,b]$) is either a section of K_1 or a section of K_2 and so Proposition 2.14(b) implies that $T(K_3) \subset T(K_1) \vee T(K_2)$. Thus $T(K_3) = T(K_1) \vee T(K_2)$, and in particular, $T([a,b]) = T([0,a]) \vee T([0,b])$, as we know that $T([0,a])$ and $T([0,b])$ are in opposite directions.

The fact that the same holds for T^{-1} is proven in the same way, as we already established that T and T^{-1} share all the properties which were used in the proof above. □

Next, we shall show that, on segments, T (and T^{-1}) is order preserving.

Lemma 2.18. *Let $I, S \in \mathcal{K}_0^n$ be two segments. Then, $I \subset S \Leftrightarrow T(I) \subset T(S)$.*

Proof. By Proposition 2.2, T preserves the dimension of any set $K \in \mathcal{K}_0^n$. Hence, by Lemma 2.17, it is enough to assume that I and S are decent segments and so we denote $I = [0,a]$ and $S = [0,b]$ (where $[0,b]$ may also be a ray). Assume that $I \subset S$ but $T(I) \not\subset T(S)$. Since $T([0,a]) \not\subset T([0,b])$, Lemma 2.16 implies that $T([0,b]) \subsetneqq T([0,a])$. Let $[0,x] \in \mathcal{K}_0^n$ be a decent segment which is linearly independent of $[0,a]$. Then, by Lemma 2.4, $T([0,x])$ is linearly independent of $T([0,a])$ (and of $T([0,b])$). Let M be the pre-image of the triangle $T([0,x]) \vee T([0,b])$. Then, M is a 2-dimensional set which, by Proposition 2.14(b), has both $[0,x]$ and $[0,b]$ as its sections. Let $[0,y] \subset [0,x] \vee [0,b]$ be a section of M (may be a ray) which is different from both $[0,x]$ and $[0,b]$. Thus, $T([0,y])$ is a section of $T([0,x]) \vee T([0,a])$, where $T([0,y]) \neq T([0,x])$ and $T([0,y]) \neq T([0,a])$ (by injectivity). Now, define the set $K = [0,a] \vee [0,y] \vee [0,x]$. Since T maps any section of K to a section of $T(K)$, it follows that $T([0,y])$, $T([0,x])$ and $T([0,a])$ are all sections of $T(K)$. But, $T([0,y])$ cannot be a section of $T(K)$ since $T([0,y]) \subset T([0,x]) \vee T([0,b])$ whereas $T(K) \supset T([0,x]) \vee T([0,a]) \supsetneq T([0,x]) \vee T([0,b])$, as we assumed that $T([0,a]) \supsetneq T([0,b])$, a contradiction. Thus, $T([0,a]) \subset T([0,b])$. The fact that the same holds for T^{-1} is proven in the same way. $\square$

We are now ready to conclude that T is an order-isomorphism. That is,

Lemma 2.19. *Let $K, L \in \mathcal{K}_0^n$. Then, $K \subset L \Leftrightarrow T(K) \subset T(L)$.*

Proof. For any two sets $K, L \in \mathcal{K}_0^n$, we have that $K \subset L$ if and only if for each 1-dimensional subspace E, $K \cap E \subset L \cap E$ which, by Lemma 2.18 (together with Lemma 2.1, in case $\mathcal{K} \cap E = \{0\}$) holds if and only if $T(K \cap E) \subset T(L \cap E)$. By Proposition 2.14(b) this holds if and only if $T(K) \cap T(E) \subset T(L) \cap T(E)$, where $T(E)$ is a 1-dimensional subspace as well. Since, by Lemma 2.5, T is a bijective mapping on the 1-dimensional subspaces, it follows that $T(K) \cap T(E) \subset T(L) \cap T(E)$ if and only if $T(K) \subset T(L)$. $\square$

As already mentioned, by Theorem 1.4, T is induced by a linear point map, which completes our proof.

3. Proof of Theorem 1.5

Obviously, if $n \geqslant 3$, the theorem holds as we already showed a stronger version of it, in Theorem 1.6. Thus we are only left with the 2-dimensional case. As mentioned in Remark 2.13, every step of the proof of Theorem 1.6, except Lemma 2.12, holds in the 2-dimensional case as well. Thus, if we show that under our assumptions Lemma 2.12 holds for $n = 2$, the rest of the proof can be repeated verbatim.

Lemma 3.1. *Let $K \in \mathcal{K}_0^2$ be a 2-dimensional set, and let $E \subset \mathbb{R}^2$ be a 1-dimensional subspace such that $K \cap E = 0$. Then, $T(K) \cap T(E) = 0$.*

Proof. Assume that the claim is false. Then $L := T(K) \cap T(E)$ is a 1-dimensional set. Applying properties of T^{-1} we have $T^{-1}(L) = T^{-1}(T(K) \cap T(E)) = T^{-1}(T(K)) \cap F = K \cap F$. But, we already know by Lemma 5.4 that pre-images of linearly dependent segments are linearly dependent, hence $T^{-1}(L) \subset E$, which is a contradiction to the fact that $K \cap E = 0$. This completes the proof. $\square$

4. Proof of Theorem 1.7

Some steps of the proof resemble those of Theorem 1.6, hence details of such steps will be omitted.

The plan of the proof is as follows. First we show that such T preserves dimension. Then we check the theorem's hypothesis implies the following: *For every 1-dimensional subspace $E \subset \mathbb{R}^n$, there exists a 1-dimensional subspace $F \subset \mathbb{R}^n$ such that for every $K \in \mathcal{K}_c^n$ we have* $T(K \cap E) = T(K) \cap F$. We will use the latter result to conclude that pre-images of 1-dimensional sections are 1-dimensional sections – a conclusion that will lead us to the fact that T is induced by its restriction to 1-dimensional sets. To finish the proof we will show that T is an order-isomorphism and using Gruber's results we will conclude that T is induced by a linear map.

4.1. Fixed origin

Lemma 4.1. $T(\{0\}) = \{0\}$.

Proof. Same as the proof of Lemma 2.1. □

4.2. Dimension invariance

Proposition 4.2. *T preserves dimensions, i.e.,* $\dim K = \dim T(K)$ *for every* $K \in \mathcal{K}_c^n$.

Proof. Same as the proof of Proposition 2.2. □

4.3. Subspace invariance

As in Theorem 1.6, we would like to show that sections by a 1-dimensional subspace E are mapped to sections by a subspace denoted by E' (also 1-dimensional by Proposition 4.2), which depends only on E. To this end, we will first discuss 2-dimensional sections. For a set $K \in \mathcal{K}_c^n$ and a subspace E, denote the subspace F that satisfies $T(K \cap E) = T(K) \cap F$ by $F_{E,K}$.

Lemma 4.3. *Let $S_1, S_2 \in \mathcal{K}_c^n$ be linearly independent segments. Then $T(S_1), T(S_2)$ are also linearly independent segments.*

Proof. Same as the proof of Lemma 2.4. □

Lemma 4.4. *Let E be a* 2*-dimensional subspace and let $K_1, K_2 \in \mathcal{K}_c^n$ be* 2*-dimensional sets contained in E, such that $\partial K_1 \cap \partial K_2$ has at least* 4 *points. Then $F_{E,K_1} = F_{E,K_2}$.*

Proof. Since 0 is in the relative interior of both sets, the set $K := K_1 \cap K_2$ is 2-dimensional and symmetric. Since ∂K has at least 4 points, we can choose four of them to be $-x_1, x_1, -x_2, x_2$. Then, K has two sections $S_1 = [-x_1, x_1]$ and $S_2 = [-x_2, x_2]$. Obviously S_1, S_2 are sections of both K_1 and K_2, and they are mapped to sections of $T(K_1)$ and $T(K_2)$. By Proposition 4.2, $\dim F_{E,K_1} = \dim F_{E,K_2} = 2$. By Lemma 4.3, $T(S_1), T(S_2)$ are linearly independent, hence they must span both F_{E,K_1} and F_{E,K_2}, which completes the proof. □

Lemma 4.5. *Let E be a* 2*-dimensional subspace and let $K_1, K_2 \in \mathcal{K}_c^n$ be* 2*-dimensional sets contained in E, then $F_{E,K_1} = F_{E,K_2}$.*

Proof. If $\partial K_1 \cap \partial K_2$ has at least 4 points, according to the previous lemma we are done. If not, then we know that either $K_1 \subset K_2$ or $K_2 \subset K_1$. Assume without loss of generality that $K_1 \subset K_2$. Choose some centrally symmetric set $K_3 \subset E$ such that $\partial K_1 \cap \partial K_3$ and $\partial K_3 \cap \partial K_2$ have at least 4 points, each. Using Lemma 4.4, we have $F_{E,K_1} = F_{E,K_3}$ and $F_{E,K_2} = F_{E,K_3}$, hence $F_{E,K_1} = F_{E,K_2}$. □

Corollary 4.6. *From Lemma* 4.5 *it follows that for every* 2*-dimensional subspace E there exists a* 2*-dimensional subspace F which depends only on E, such that for any $K \in \mathcal{K}_c^n$, for which* $\dim(K \cap E) = 2$*, we have $T(K \cap E) = T(K) \cap F$. The subspace corresponding to E will be denoted by E'.*

Lemma 4.7. *Let L be a* 2*-dimensional set in $\mathcal{K}_c^n$, and let $E_0 \subset \mathbb{R}^n$ be a* 2*-dimensional subspace, such that* $\dim(L \cap E_0) = 1$. *Then $T(L \cap E_0) = T(L) \cap E_0'$.*

Proof. Denote $s = L \cap E_0$ and choose 2-dimensional set $K \subset E_0 \cap K_c^n$, such that s is a section of K. Notice that by Lemma 4.5, $T(s) \subset T(K) \subset E_0'$ and $T(s) \subset T(L)$, hence $T(s) \subset T(L) \cap E_0'$. We will show that $T(s) = T(L) \cap E_0'$. Indeed, assume that $\dim T(L) \cap E_0' = 2$, that is, $T(L) \cap E_0' = T(L)$. Consider the set $K \vee L$. Then, on the one-hand $T((K \vee L) \cap E_0) = T(K) = T(K \vee L) \cap E_0'$, and on the other hand, $T((K \vee L) \cap \operatorname{sp} L) = T(L) = T(K \vee L) \cap E_0'$, which is a contradiction to the injectivity of T. $\square$

We are ready to show the following result.

Lemma 4.8. *Pre-images of linearly independent segments are linearly independent segments.*

Proof. Assume we have a 2-dimensional set K such that S is a section of K. Then, there exists a 2-dimensional subspace E, such that $T(S) = T(K \cap E) = T(K) \cap E'$, where the last equality is due to Lemma 4.7. This means that $T(S)$ is contained in E'. Similarly, we take another 2-dimensional set M, such that $K \cap M = S$, and the corresponding 2-dimensional subspace F such that $T(S) = T(M \cap F) = T(M) \cap F'$, which implies that $T(S) \subset F'$. Notice that here we used the fact that the dimension is greater or equal to 3. Due to Lemma 4.7, F' does not depend on M and E' does not depend on K, and so $E' \cap F'$ is a 1-dimensional subspace containing $T(S)$. The same argument applies to any segment $L \subset \operatorname{sp} S$, thus any such segment L will be mapped to a segment in the same 1-dimensional subspace $E' \cap F'$. $\square$

4.4. Zero intersection

Lemma 4.9. *Let $K \in \mathcal{K}_c^n$ and let $S \in \mathcal{K}_c^n$ be a segment such that $K \cap S = 0$. Then, $T(K) \cap T(S) = 0$.*

Proof. Denote the relative interior of a set K by *relint* K. Notice that if $K \cap S = 0$, then $\operatorname{sp} K \cap S = 0$ (since $0 \in \mathit{relint}\, K$), so $\dim K \leqslant n - 1$. Assume that $T(K) \cap T(S) \neq 0$. Without loss of generality we may assume that $T(S)$ is not a section of $T(K)$. Otherwise we could choose $S' \subset S$, and the proof would not change, as by Lemma 4.8, we know that $T(S') \subset \operatorname{sp} T(S)$. Define the cylinder $\tilde{K} = K \vee S$. Then $T(\tilde{K} \cap \operatorname{sp} S) = T(S)$ and $T(\tilde{K} \cap \operatorname{sp} K) = T(K)$ are sections of $T(\tilde{K})$. But this cannot happen since $T(S) \subset T(K)$, and $T(S)$ is not a section of $T(K)$. $\square$

Corollary 4.10. *The pre-image of every* 1*-dimensional section of a set $K \in \mathcal{K}_c^n$ is a section of $T^{-1}(K)$.*

To see this, notice that the pre-image of any 1-dimensional section S of a set $K \in \mathcal{K}_c^n$ is either a section of $T^{-1}(K)$ or satisfies that $T^{-1}(S) \cap T^{-1}(K) = 0$. However, the latter cannot hold due to Lemma 4.9.

Corollary 4.11. *For every* 1*-dimensional subspace E, there exists a* 1*-dimensional subspace F such that for every set $K \in \mathcal{K}_c^n$ we have $T(K \cap E) = T(K) \cap F$.*

The corollary follows trivially by checking two cases. If $K \cap E = 0$, the claim follows from Lemma 4.9. If $\dim(K \cap E) = 1$, the claim holds due to Lemma 4.7.

4.5. Order-isomorphism

Lemma 4.12. *Let $K_1, K_2 \in \mathcal{K}_c^n$ be* 2*-dimensional sets contained in a* 2*-dimensional subspace E, that satisfy the following*:

(1) $K_1 \neq K_1 \cap K_2$.
(2) $K_2 \neq K_1 \cap K_2$.
(3) $K_1 \cup K_2 \notin \mathcal{K}_c^n$.

Then, $T(K_1 \cap K_2) = T(K_1) \cap T(K_2)$.

Proof. Denote the intersection $K_1 \cap K_2$ by K. Then, by Corollary 4.10, the pre-image of any 1-dimensional section of $T(K)$ is either a section of K_1 or a section of K_2. Hence, all 1-dimensional sections of $T(K)$ are sections of either $T(K_1)$ or of $T(K_2)$. As 1-dimensional sections uniquely determine the set, $T(K)$ must be one of the following four candidates: $T(K_1), T(K_2), T(K_1) \cup T(K_2), T(K_1) \cap T(K_2)$. By injectivity, $T(K_1)$ and $T(K_2)$ are ruled out. Using the above argument for T^{-1} (note that by the previous two corollaries, T and T^{-1} have the same properties), $T^{-1}(T(K_1) \cap T(K_2))$ must be either $K_1 \cap K_2$ or $K_1 \cup K_2$. Since the latter is not convex, we conclude that $T^{-1}(T(K_1) \cap T(K_2)) = K_1 \cap K_2$, as required. □

Lemma 4.13. *Let $S_1 \subsetneq S_2$ be two segments in $\mathcal{K}_c^n$. Then $T(S_1) \subsetneq T(S_2)$.*

Proof. Choose two 2-dimensional sets K_1, K_2 in a 2-dimensional subspace E, such that S_1 is a section of K_1, S_2 is a section of K_2 and K_1, K_2 satisfy conditions (1)–(3) of Lemma 4.12. Then, applying Lemma 4.12, we have $T(K_1 \cap K_2) = T(K_1) \cap T(K_2)$. This implies that $T(S_1)$ is a section of $T(K_1) \cap T(K_2)$. Since $T(S_2)$ is a section of $T(K_2)$, and since by Lemma 4.8 $T(S_1)$ and $T(S_2)$ are linearly dependent, it follows that $T(S_1) \subset T(S_2)$. □

Lemma 4.14. *Let $S_1 \subsetneq S_2$ be two segments in $\mathcal{K}_c^n$. Then $T^{-1}(S_1) \subsetneq T^{-1}(S_2)$.*

Proof. By Lemma 4.3, we know that $T^{-1}(S_1)$ and $T^{-1}(S_2)$ belong to the same 1-dimensional subspace. As we deal only with symmetric sets this means that either $T^{-1}(S_1) \subset T^{-1}(S_2)$ or $T^{-1}(S_2) \subset T^{-1}(S_1)$, and so Lemma 4.13 implies that $T^{-1}(S_1) \subset T^{-1}(S_2)$. □

Next, notice that Lemma 4.13 together with Lemma 4.14 imply that T is an order-isomorphism. By Theorem 1.4, we know that there exists $G_0 \in GL(n)$, such that restriction of T to sets in $\mathcal{K}_c^n$ with non-empty interior is induced by G_0. That is, for any $K \in \mathcal{K}_c^n$ such that $\dim K = n$, we have that $T(K) = G_0 K$. In order to complete our proof we need to check that $T(K) = G_0 K$ for any $K \in \mathcal{K}_c^n$. First, we shall verify this claim for 1-dimensional sets.

Lemma 4.15. *Let $S \in \mathcal{K}_c^n$ be a* 1*-dimensional set. Then $T(S) = G_0 S$.*

Proof. We know that for every n-dimensional set $M \in \mathcal{K}_c^n$, $T(K) = G_0 M$. Assume we have a segment $S \in \mathcal{K}_c^n$. Then, there exists a sequence $\{K_m\}_1^\infty \subset \mathcal{K}_c^n$ such that the following hold.

(1) $\dim K_m = n$ for every $m \in \mathbb{N}$.
(2) S is a section of K_m for every $m \in \mathbb{N}$.
(3) $K_m \searrow S$, that is, the sequence satisfies $K_i \supset K_{i+1}$ for all i and $T(S) = \bigcap_{i \in \mathbb{N}} K_m$.

Define $E := \mathrm{sp}\{S\}$. By Corollary 4.11, there exists a 1-dimensional subspace E' which depends only on E, such that $T(S) = T(K_n \cap E) = T(K_n) \cap E' = G_0 K_n \cap E'$. Thus, as K_n converges to S we get that $T(S) = G_0 S \cap E'$. Since $G_0 S \cap E' \neq 0$, it follows that $G_0 S \subset E'$ and so $T(S) = G_0 S$. $\square$

To conclude the proof, one should only notice that for any set $K \in \mathcal{K}_c^n$, we have that $T(K) = \bigcup_{E \in G_{n,1}} T(K \cap E)$, where $G_{n,1}$ is the Grassmannian of lines through the origin in $\mathbb{R}^n$. Indeed, it is clear that $\bigcup_{E \in G_{n,1}} T(K \cap E) \subset T(K)$ since T maps sections to sections. To show the inclusion in the other direction, note that any point in $T(K)$ belongs to a 1-dimensional section of $T(K)$, say $T(K) \cap F$ (where F is the appropriate 1-dimensional subspace) which, by Corollary 4.10, is the image of a 1-dimensional section of K, that is $T(K) \cap F = T(K \cap E)$ for some $E \in G_{n,1}$. Hence $T(K) \subset \bigcup_{E \in G_{n,1}} T(K \cap E)$. Finally, $T(K) = \bigcup_{E \in G_{n,1}} T(K \cap E) = A \bigcup_{E \in G_{n,1}} (K \cap E) = AK$.

5. Proof of Theorem 1.9

The plan of the proof is as follows. We show that conditions (a) and (c) of Theorem 1.9 imply that T preserves dimension. Then we conclude that condition (b) combined with several lemmas imply that 1-dimensional sections of sets are mapped to 1-dimensional sections of their images under T and T^{-1}. This will suffice for the completion of the proof by using known theorems about order-isomorphisms for convex bodies.

5.1. Fixed origin

Lemma 5.1. $T(\{0\}) = \{0\}$.

Proof. Same as the proof of Lemma 2.1. $\square$

5.2. Dimension invariance

Proposition 5.2. *For every $K \in \mathcal{K}_0^n$, we have* $\dim K = \dim T(K) = \dim T^{-1}(K)$.

The proof is composed of several lemmas.

Lemma 5.3. *For every $K \in \mathcal{K}_0^n$ such that* $\dim K = 1$ *it follows that* $\dim T(K) = \dim T^{-1}(K) = 1$.

Proof. Assume we have a 1-dimensional set S, such that $T(S)$ is m-dimensional, where $m > 1$. Take two 1-dimensional linearly independent sections E_1, E_2, of the set $T(S)$. By property (a) of T^{-1}, their pre-images are subsets of S, hence not linearly independent. A contradiction to property (c). Since T and T^{-1} share the same properties, it follows that $\dim T^{-1}(S) = 1$ as well. $\square$

Lemma 5.4. *T preserves linear independence. That is, n linearly independent segments are mapped to n linearly independent segments.*

Proof. The proof is by induction. By property (c) and Lemma 5.3, the claim is correct for 2 linearly independent segments, so we have the induction base. Let us check that the claim holds for m linearly independent segments assuming it is true for any $k < m$. Take m linearly independent segments, $s_1, \dots, s_m$. We know that the first $m-1$ segments are mapped $m-1$ linearly independent segments $T(s_1), \dots, T(s_{m-1})$. Since we care only about linear independence, without loss of generality and by using property (c) again, we may choose s_i such that $T(s_i)$ contains 0 in its relative interior (for any $1 \leqslant i \leqslant m-1$). Define the set $K := s_1 \vee s_2 \vee \cdots \vee s_{m-1}$. By property (a), the set $T(K)$ contains the set $L := T(s_1) \vee T(s_2) \vee \cdots \vee T(s_{m-1})$, and by the choice of s_i we have that $0 \in \mathit{relint}\, L$. Assume $T(s_m)$ lies in $\operatorname{sp} L$. Then, since $0 \in \mathit{relint}\, L$, the intersection $(T(K) \cap \operatorname{sp} L) \cap T(s_m)$ is 1-dimensional. Thus, by property (a) $T^{-1}((T(K) \cap \operatorname{sp} L) \cap \operatorname{sp} T(s_m)) \subset K$. On the other hand, by property (c), $T^{-1}((T(K) \cap \operatorname{sp} L) \cap \operatorname{sp} T(s_m)) \subset \operatorname{sp}\{s_m\}$ which is a contradiction. $\square$

To conclude Proposition 5.2 notice that Lemma 5.4 implies that for any $K \in \mathcal{K}_0^n$ we have $\dim T(K) \geqslant \dim K$. But T and T^{-1} have the same properties, so equality must hold.

Remark 5.5. Proposition 5.2 implies that 1-dimensional sets are always mapped to 1-dimensional sets, hence property (c) holds for any two linearly independent segments. This is used throughout the proof.

5.3. Properties of segments

Lemma 5.6. *T and T^{-1} map decent segments to decent segments.*

Proof. By Proposition 5.2, segments are mapped to segments. Assume we have a decent segment L for which $T(L)$ contains a segment $[-a, a]$, where a is some point in $\mathbb{R}^n \setminus \{0\}$. Let S be a decent segment, linearly-independent of L. By property (a), the image of the (generalized) triangle $K = S \vee L$ contains $T(S) \vee T(L)$. Take some segment R so that $R \cap K = 0$. By Lemma 5.4, $T(S), T(L)$ and $T(R)$ are linearly dependent, and so our assumption (on $T(L)$) implies that $P := \operatorname{sp} T(R) \cap T(K)$ is 1-dimensional. Using Lemma 5.4 once more, we have that $T^{-1}(P)$ is linearly dependent of R. On the other hand, property (a) implies that $T^{-1}(P) \subset K$, which means that R and $T^{-1}(P)$ are linearly independent – a contradiction. $\square$

Lemma 5.7. *Let $\bar{x}, \bar{y}$, and $\bar{z}$, be three decent segments, linearly independent in pairs. Assume that $\bar{z} \cap (\bar{x} \vee \bar{y}) \neq 0$. Then, $T(\bar{z}) \cap (T(\bar{x}) \vee T(\bar{y})) \neq 0$.*

Proof. First, assume we have a decent segment $\overline{w}$, such that $K := \bar{x} \vee \bar{y}$ has 0 intersection with $\overline{w}$ and assume that $T(\overline{w}) \cap (T(\bar{x}) \vee T(\bar{y})) \neq 0$. By property (a), it follows that $T(\bar{x}) \vee T(\bar{y}) \subset T(K)$ and that $T^{-1}(T(K) \cap \operatorname{sp} T(\overline{w})) \subset K$. Since $T(\overline{w}) \cap (T(\bar{x}) \vee T(\bar{y})) \neq 0$, it follows that $T^{-1}(T(K) \cap \operatorname{sp} T(\overline{w}))$ is 1-dimensional. This is a contradiction since by property (c), $T^{-1}(T(K) \cap \operatorname{sp} T(\overline{w}))$ is a subset of $\operatorname{sp} w$, which has 0 intersection with K. Now the lemma follows, using the above argument for T^{-1} and z. $\square$

Remark 5.8. From Lemma 5.7 it follows that, if $T(\bar{x}) = \overline{x'}$, and $T(\bar{y}) = \overline{y'}$, where $\bar{x}$ and $\bar{y}$ are linearly independent, then $\overline{x'}$ and $\overline{y'}$ belong to the boundary of the image of the triangle $K = \bar{x} \vee \bar{y}$. Indeed, otherwise, we would have a section of K which T maps to a decent segment that has zero intersection with $\overline{x'} \vee \overline{y'}$ and this would contradict Lemma 5.7.

Lemma 5.9. *T maps finite decent segments to finite decent segments, and rays to rays.*

Proof. By Lemma 5.6, a finite decent segment is mapped either to a ray or to a finite decent segment. We prove that it cannot be mapped to a ray. Since T and T^{-1} share the same properties, this will suffice.

First assume there exist two linearly independent finite decent segments $\bar{x}, \bar{y}$ that are mapped to rays r_u, r_v, respectively. Then, due to Remark 5.8, the image of the triangle $\bar{x} \vee \bar{y}$ must be $r_u \vee r_v$. However, in this case we could define a convex set K with $\bar{x}, \bar{y}$ as its sections, which is different from $\bar{x} \vee \bar{y}$, and get that also $T(K) = r_u \vee r_v$, which would contradict the injectivity of T.

Next, assume there is a finite decent segment s which is mapped to a ray $T(s)$, whereas any finite segment linearly independent of s is not mapped to a ray. Let r_u be a decent segment (by assumption, it must be a ray) which is mapped to a ray $T(r_u)$. According to Remark 5.8, the set $K := s \vee r_u$ is mapped to the cone $C := T(r_u) \vee T(s)$. Take a 1-dimensional ray r, different from both $T(r_u)$ and $T(s)$, that is a section of C. The pre-image of r is contained in K (by property (a)), and so must be finite. This contradicts the assumption that there are no finite segments linearly independent of s that are mapped to rays. $\square$

Lemma 5.10. *T maps 1-dimensional subspaces to 1-dimensional subspaces.*

Proof. Let L be a segment, which is not a subspace, and assume that $T(L)$ is a 1-dimensional subspace. By Lemma 5.6, L is not a decent segment. Take another segment S, linearly independent of L so that $T(S)$ is a subspace (linearly independent of $T(S)$, by Lemma 5.4). Then, by property (a) and Proposition 5.2, it follows that $U := T(S \vee L)$ is a 2-dimensional subspace. Since all 1-dimensional sections of U are subspaces, and since (by property (a)) their pre-images are subsets of $S \vee L$, it follows that one may find a 1-dimensional set K for which $K \vee L \subsetneq S \vee L$ but $T(K \vee L)$ is the same 2-dimensional subspace U – a contradiction to the injectivity of T. $\square$

5.4. Sections are mapped to sections

A 2-dimensional body which the closure of the convex hull of a 1-dimensional subspace and a decent segment will be called a *decent slab*. Recall that sets $K_1, K_2 \in \mathcal{K}_0^n$ are said to be *comparable* if either $K_1 \subset K_2$ or $K_2 \subset K_1$. An immediate consequence of Lemmas 5.10, 5.6 is the fact that images of decent slabs are decent slabs.

Notice the following fact:

Remark 5.11. Comparable decent slabs are mapped to comparable decent slabs.

Indeed, if the images of two comparable slabs S_1, S_2 were not comparable, we could find a 1-dimensional section s of $T(S_1)$ that has 0 intersection with any section of $T(S_2)$. But then, by property (c), $T^{-1}(s)$ would have 0 intersection with any section of S_2 which cannot happen as S_1 and S_2 are comparable.

Now we would like to check monotonicity for segments in every direction and this way to show that T is an order-isomorphism. This is the point where property (b) plays a role. First, we show the following fact:

Lemma 5.12. *Let S be a decent slab, for which $\overline{\lambda x_0}$ is a 1-dimensional section of S for some $\lambda > 0$. Then, $T(\overline{\lambda x_0})$ is a section of $T(S)$.*

Proof. Assume this is not true. Then, we can extend the decent segment $T(\overline{\lambda x_0})$ to some decent segment L that is a section of $T(S)$ (L must be decent since Lemma 5.6 applies to T^{-1} as well). By property (a) for T^{-1} we have that $T^{-1}(L) \subsetneq \overline{\lambda x_0}$ which contradicts property (b). $\square$

Now, we may use the preceding argument to show monotonicity for decent segments in every direction:

Lemma 5.13. *For every $y \in \mathbb{R}^n$ and for every $\lambda_1, \lambda_2 > 0$, $\overline{\lambda_1 y} \subset \overline{\lambda_2 y}$ implies $T(\overline{\lambda_1 y}) \subset T(\overline{\lambda_2 y})$.*

Proof. Take some point $y \in \mathbb{R}^n$ such that $y \notin span\{x_0\}$. Assume $z = \lambda y$ for $0 < \lambda < 1$. Let S_1 be the slab defined by taking the convex hull of the 1-dimensional subspace E parallel to the line through y and x_0, with x_0. Now, take the slab S_2, comparable to S_1, with section $\bar{z}$. Notice that λx_0 is a section of S_2. Due to Lemma 5.12, $T(\overline{x_0})$ and $T(\overline{\lambda x_0})$ are sections of $T(S_1)$ and $T(S_2)$ respectively, and hence $T(S_2) \subset T(S_1)$ (for any $0 < \lambda < 1$), where we used the fact that $T(\overline{\lambda x_0}) \subset T(\overline{x_0})$, by property (b).

We need to show that $T(\bar{y})$ is a section of $T(S_1)$. Assume it is not, then $T(\bar{y})$ can be extended to some decent segment L which is a section of $T(S_1)$. Applying properties of T^{-1} we have that $l := T^{-1}(L) \subsetneq \bar{y}$. Choose λ so that $\bar{z} = l$. Using the above argument $S_2 \subset S_1$, hence $T(S_2) \subset T(S_1)$, which is a contradiction since $T(l)$ is not contained in $T(S_2)$. $\square$

Lemma 5.14. *Let $K \in \mathcal{K}_0^n$ such that $0 \notin relint\, K$. Then $0 \notin relint\, T(K)$.*

Proof. We already know that the claim holds for decent segments, so it is enough to check the case where $\dim K > 1$. Assume $0 \in relint\, T(K)$. Take a 1-dimensional section S of K. Since $0 \notin relint\, K$, S can be chosen to be a decent segment and by Lemma 5.6, $T(S)$ is also decent. By property (a), $T(S) \subset T(K)$ and as it is a decent segment it can be extended to a non-decent one, L, which is a section of $T(K)$. Applying property (a) for T^{-1} we have that $T^{-1}(L) \subset K$. But $T^{-1}(L)$ is in $\mathrm{sp}\, S$, hence $T^{-1}(L) \subset S$, which means it is a decent segment. This is a contradiction to Lemma 5.6. $\square$

Corollary 5.15. *Let $K \in \mathcal{K}_0^n$ have a section S which is a decent segment. Then the decent segment $T(S)$ $(T^{-1}(S))$ is a section of $T(K)$ $(T^{-1}(K))$.*

Proof. If $T(S)$ were not a section of $T(K)$, we could extend it to some decent segment L which is a section of $T(K)$. But then, by property (a), $T^{-1}(L)$ must be contained in S and this is a contradiction to monotonicity shown in Lemma 5.13. $\square$

Summarizing the above, we showed that sections which are decent segments are mapped to sections (which are again decent).

Lemma 5.16. *Let $[0,a]$ and $[0,b]$ be segments in opposite directions. Then, $T([0,a])$ and $T([0,b])$ are in opposite directions. Similarly, $T^{-1}([0,a])$ and $T^{-1}([0,b])$ are in opposite directions.*

Proof. The proof of Lemma 2.16 works in this case due to Corollary 5.15. $\square$

It is left to check how non-decent sections behave under T. First we show that a segment which is a union of two decent segments is mapped to the union of their images:

Lemma 5.17. *Let $[a,b] \in \mathcal{K}_0^n$ be a segment of the form $[a,b] = [0,a] \vee [0,b]$. Then $T([a,b]) = T([0,a]) \vee T([0,b])$ (and so $T^{-1}([a,b]) = T^{-1}([0,a]) \vee T^{-1}([0,b])$).*

Proof. Once again, the proof of Lemma 2.17 holds in this case due to Corollary 5.15. $\square$

Combining Lemmas 5.17 and 5.13 we come to the following:

Corollary 5.18. *Let $S_1, S_2 \in \mathcal{K}_0^n$ be 1-dimensional segments. Then $S_1 \subset S_2 \Leftrightarrow T(S_1) \subset T(S_2)$.*

Lemma 5.19. *Let S be a 1-dimensional section of $K \in \mathcal{K}_0^n$. Then $T(S)$ $(T^{-1}(S))$ is a section of $T(K)$ $(T^{-1}(K))$.*

Proof. Assume that $T(S)$ is not a section of $T(K)$. Then the section $L := T(K) \cap \mathrm{sp}\{T(S)\}$ contains $T(S)$. By Corollary 5.18 $S \subset T^{-1}(L)$. On the other hand by property (a), $T^{-1}(L) \subset K$ which actually means that $T^{-1}(L) \subset S$ (since linearly dependent segments are mapped to linearly dependent segments), a contradiction. $\square$

We are now ready to conclude that T is an order-isomorphism. That is,

Lemma 5.20. *Let $K, L \in \mathcal{K}_0^n$. Then, $K \subset L \Leftrightarrow T(K) \subset T(L)$.*

Proof. $K \subset L$ if and only if for each 1-dimensional subspace E, $K \cap E \subset L \cap E$ which, by Corollary 5.18 and the previous step, holds if and only if $T(K) \cap T(E) \subset T(L) \cap T(E)$. $\square$

As in Theorem 1.6 this completes our proof.

Acknowledgments

The authors would like to thank Prof. Shiri Artstein-Avidan for fruitful discussions. In addition the authors would like to thank the Fields Institute for its hospitality while this research was conducted.

Appendix A. The Grassmannian

In this section, we prove the following fact, which has the same flavor as our main results in this paper, but does not formally follow from them. Its proof is in the same spirit as previous proofs in this note. However, we find this result interesting enough to be added in this appendix.

Let $G_{n,k}$ be the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$ and consider the union of all Grassmannians, $\mathcal{G}^n := \bigcup_{k=0}^n G_{n,k}$, where for convenience we let the point 0 be a 0-dimensional Grassmannian. For $g_1, g_2 \in \mathcal{G}^n$ we will write $g_1 \hookrightarrow g_2$ whenever g_1 is a subspace of g_2. We prove the following:

Theorem A.1. *Let $n \geqslant 3$. Let $T : \mathcal{G}^n \to \mathcal{G}^n$ be a bijective map and assume that T is order preserving, that is $g_1 \hookrightarrow g_2$ implies $T(g_1) \hookrightarrow T(g_2)$, for any $g_1, g_2 \in \mathcal{G}^n$. Then, there exists a linear transformation $A \in GL(n)$ such that $T(g) = Ag$, for every $g \in \mathcal{G}^n$.*

Proof. *Fixed origin*: Since T preserves order and 0 is contained in all subspaces, it follows that $T(0)$ is contained in all subspaces and so $T(0) = 0$.

Dimension preservation: Here we will show that $\dim T(g) = \dim g$ for every $g \in \mathcal{G}^n$. First, let us check that $\dim T(g) \geqslant \dim g$, for every $g \in \mathcal{G}^n$. We know that 0 is mapped to 0, hence 1-dimensional subspaces are mapped to subspaces of dimension at least 1. Assume that the claim holds for any dimension up to some m and assume that there exists some $(m+1)$-dimensional subspace g such that $\dim T(g) = p < m+1$. Take $h \hookrightarrow g$ of dimension p. By the induction hypothesis we have that $\dim T(h) \geqslant p$ and by the order preserving property of T it follows that $T(h) \hookrightarrow T(g)$. Thus $\dim T(h) = p$ and consequently $T(h) = T(g)$, which is a contradiction to the injectivity of T.

Next we will check that, indeed, T preserves dimension. By the last step, we have that $T(\mathbb{R}^n) = \mathbb{R}^n$. Inductively, we assume that T preserves the dimension of subspaces of dimension $> m$ and show that it preserves the dimension of m-dimensional subspaces. Let g be an m-dimensional subspace and assume that $k = \dim T(g) > m$ (by the previous step, it cannot be that $\dim T(g) < m$). Then, there exists a k-dimensional subspace h containing g and so $T(g) \hookrightarrow T(h)$. By the induction hypothesis, $\dim T(h) = k$ and so it follows that $T(g) = T(h)$, a contradiction to injectivity.

Linear independence: Next, we show that for any k linearly independent 1-dimensional subspaces $h_1, \dots, h_k$, we have that $T(h_1), \dots, T(h_k)$ are linearly independent as well. The proof is by induction on k. Obviously the claim holds for $k = 2$ due to injectivity. Assume that the claim holds for $k < m$ and let $h_1, \dots, h_m$ be m linearly independent 1-dimensional subspaces. Denote the span of $h_1, \dots, h_{m-1}$ by H^1_{m-1} and the span of $h_2, \dots, h_m$ by H^2_{m-1}. By the previous step, we have that both $T(H^1_{m-1})$ and $T(H^2_{m-1})$ are $(m-1)$-dimensional subspaces. By the induction hypothesis, and due to the order preserving property of T, it follows that $T(h_1), \dots, T(h_{m-1})$ span the $(m-1)$-dimensional subspace $T(H^1_{m-1})$ while $T(h_2), \dots, T(h_m)$ span the $(m-1)$-dimensional subspace $T(H^2_{m-1})$. Altogether, we have that $T(h_1), \dots, T(h_m)$ span two different $(m-1)$-dimensional subspaces, and so they are linearly independent.

Inducing linear map: Consider the restriction of T to $G_{n,1}$, denoted by f. By assumption, f is injective. Moreover, since T is order preserving and also preserves dimension, it follows that f maps any three coplanar 1-dimensional subspaces to coplanar 1-dimensional subspaces. Thus, by the fundamental theorem of projective geometry (see [5] for an elementary proof) there exists a linear transformation $A \in GL(n)$ such that $T(g) = Ag$ for all 1-dimensional subspaces $g \in \mathcal{G}^n$. In fact, A induces T entirely. Indeed, let $g \in \mathcal{G}^n$ with $m = \dim g$ and let $h_1, \dots, h_m$ be 1-dimensional subspaces spanning g. By the previous step, we have that $T(h_1), \dots, T(h_m)$ span $T(g)$. Thus $T(g) = \mathrm{sp}\{T(h_1), \dots, T(h_m)\} = \mathrm{sp}\{Ah_1, \dots, Ah_m\} = Ag$. This concludes our proof. $\square$

Remark A.2. Notice that the requirement in Theorem A.1 that $n \geqslant 3$ is trivially necessary, since the theorem is false in the 2-dimensional case. Indeed for $n = 2$, we may map elements of $G_{n,1}$ to elements of $G_{n,1}$ bijectively in many non-linear ways.

References

[1] S. Artstein-Avidan, V. Milman, The concept of duality in convex analysis, and the characterization of the Legendre transform, Ann. Math. 169 (2009) 661–674.
[2] S. Artstein-Avidan, V. Milman, The concept of duality for measure projections of convex bodies, J. Funct. Anal. 254 (2008) 2648–2666.
[3] S. Artstein-Avidan, V. Milman, A characterization of the concept of duality, Electron. Res. Announc. Math. Sci. 14 (2007) 48–65.
[4] K. Böröczky, R. Schneider, A characterization of the duality mapping for convex bodies, Geom. Funct. Anal. 18 (2008) 657–667.
[5] C.A. Faure, An elementary proof of the fundamental theorem of projective geometry, Geom. Dedicata 90 (2002) 145–151.
[6] P. Gruber, The endomorphisms of the lattice of norms in finite dimensions, Abh. Math. Sem. Univ. Hamburg 62 (1992) 179–189.
[7] R. Schneider, The endomorphisms of the lattice of closed convex cones, Beitrage Algebra Geom. 49 (2008) 541–547.
[8] B.A. Slomka, On duality and endomorphisms of lattices of closed convex sets, Adv. Geom. 11 (2) (2011) 225–239.